Reference Answer of Lecture 1

1. The map $\|\cdot\|: E \to \mathbb{R}$ is continuous

Proof:

$\forall x_1, x_2 \in E \quad \big|\|x_1\| - \|x_2\|\big| \le \|x_1 - x_2\|$

$\forall \varepsilon > 0, let\ \delta = \varepsilon, when\ \|x_1 - x_2\| < \delta$ we have $\big|\|x_1\| - \|x_2\|\big| \le \|x_1 - x_2\| < \delta = \varepsilon$

So the map is continuous.

2. Some properties which are true in finite dimensional Banach spaces are not necessarily true in infinite dimensional Banach spaces! For example:

(1) The closed ball $\{x \in E; \|x\| \le 1\}$ is not necessarily compact!

(2) Two norms on an infinite dimensional Banach space are not always equivalent!

(3) The Bolzano-Weierstrass theorem which says each bounded sequence has a convergent subsequence is not necessarily true!

(4) "K is compact $\Leftrightarrow$ K is both closed and bounded" is not necessarily true in a Banach space!

Counter Example: $f_n(t) = \begin{cases} 1 & (\frac{1}{n+1}, \frac{1}{n}] \\ 0 & \text{else} \end{cases}$ n=1,2,$\cdots$

$f_n(t)$ is bounded on $[0,1]$, i.e. $f_n(t) \in \mathrm{B}([0,1])$; Since $\|f_n - f_m\|_\infty = 1$ for any n, m, $\{f_n\}$ is not Cauchy, which yields $\{f_n\}$ divergent. Hence, the bounded sequence $\{f_n\}$ has no convergent sub-sequence.